Multiple Choice

1 C) $y=\sqrt{|x|}$

2 C) $P'(x)=5x^4-3x^2-2x$, both $P(1)=P'(1)=0$

3 A) Circle has $e=0$, ellipse has $e<1$, parabola has $e=1$, hyperbola has $e>1$.

4 D) $w-z$

5 B) $\dfrac{1-i}{1+i}=\dfrac{(1-i)^2}{2}=-i,\therefore$ rotation clockwise by $\dfrac{\pi}{2}$

6 C) ${}^{12}C_8+{}^{11}C_8+{}^{10}C_8+{}^{9}C_8+{}^{8}C_8=715$

7 D) $xy=c^2$ and $x^2-y^2=a^2$ are the same if $a=\sqrt{2}c$

$\sqrt{k}=\sqrt{2}\sqrt{8},\therefore k=16$

8 C) resolving the forces

9 A) $\dfrac{x-2}{6}=\dfrac{h}{4},\therefore x=\dfrac{3h}{2}+2$ and $\dfrac{y-3}{4}=\dfrac{h}{4},\therefore y=h+3$

10 B) $x+\dfrac{1}{x}=-1,\therefore x^2+x+1=0$

$\therefore x^3=1$ since $x^3-1=(x-1)(x^2+x+1)=0, x\neq 1$

$\therefore x^{2016}=1$

$\therefore x^{2016}+\dfrac{1}{x^{2016}}=2$

Question 11

(a) (i) $z=\sqrt{3}-i=2\,\text{cis}\left(-\dfrac{\pi}{6}\right)$

(ii) $z^6=2^6\,\text{cis}(-\pi)=-2^6=-64,\therefore$ real

(iii) $z^n=2^n\,\text{cis}\left(\dfrac{-n\pi}{6}\right)$ is purely imaginary when

$\cos\left(\dfrac{-n\pi}{6}\right)=0,\therefore\dfrac{n\pi}{6}=\dfrac{\pi}{2}+k\pi,\therefore n=3+6k,$

where $k\in Z$

(b) Let $u=x, dv=e^{-2x}dx$ then $du=dx, v=-\dfrac{1}{2}e^{-2x}$

$\displaystyle\int xe^{-2x}dx=-\frac{1}{2}xe^{-2x}+\frac{1}{2}\int e^{-2x}dx$

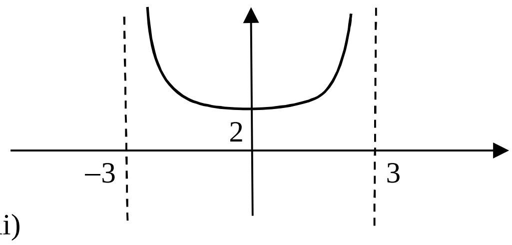
$=-\dfrac{1}{2}xe^{-2x}-\dfrac{1}{4}e^{-2x}+C$

(c) $3x^2+3y^2y'=2(y+xy')$

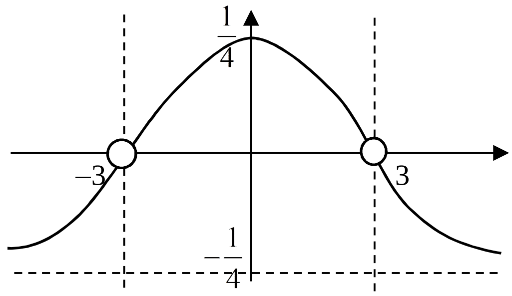
$y'\left(3y^2-2x\right)=2y-3x^2$

$y'=\dfrac{dy}{dx}=\dfrac{2y-3x^2}{3y^2-2x}$

(d) (i)



(ii)



(e) For $x\sin^{-1}\dfrac{x}{2}$, the domain is the domain of $\sin^{-1}\dfrac{x}{2}$

$\therefore$ domain is $-2\leq x\leq 2,$

When $x=2,\ f(2)=2\times\dfrac{\pi}{2}=\pi$

When $x=-2,\ f(-2)=-2\times-\dfrac{\pi}{2}=\pi$

When $x=0,\ f(0)=0$

$\therefore$ Range is $0\leq y\leq\pi$

Question 12

(a) (i) $\frac{x^2}{9}+\frac{y^2}{4}=1$

(ii) $4=9(1-e^2),\therefore e^2=1-\frac{4}{9}=\frac{5}{9},\therefore e=\frac{\sqrt{5}}{3}$

(iii) $\left(\pm\sqrt{5},0\right)$

(iv) $x=\pm\frac{9}{\sqrt{5}}$

(b) (i) $f(x)+x\ f'(x)-x\ f'(x)=f(x)$

(ii) $\therefore\int f(x)dx=x\ f(x)-\int x\ f'(x)dx$

$\therefore\int\tan^{-1}xdx=x\tan^{-1}x-\int\frac{x}{1+x^2}dx$

$=x\tan^{-1}x-\frac{1}{2}\ln(1+x^2)+C$

(c) (i) $\left(\cos\theta+i\sin\theta\right)^4=c^4+4c^3(is)+6c^2(is)^2$

$+4c(is)^3+(is)^4$, where $c=\cos\theta, s=\sin\theta$

$=\cos4\theta+i\sin4\theta$

By equating the real parts,

$\cos4\theta=\cos^4\theta-6\cos^2\theta\sin^2\theta+\sin^4\theta$

(ii) $\cos4\theta=\cos^4\theta-6\cos^2\theta(1-\cos^2\theta)+$

$(1-\cos^2\theta)^2$

$=\cos^4\theta-6\cos^2\theta+6\cos^4\theta+1-2\cos^2\theta+\cos^4\theta$

$=8\cos^4\theta-8\cos^2\theta+1$

(d) (i) $m_1=\frac{\frac{dy}{dp}}{\frac{dx}{dp}}=\frac{-\frac{c}{p^2}}{c}=-\frac{1}{p^2}$

$m_2=p^2$

$y-\frac{c}{p}=p^2(x-cp)$

$\frac{y}{p}-\frac{c}{p^2}=p(x-cp)=px-cp^2$

$px-\frac{y}{p}=c\left(p^2-\frac{1}{p^2}\right)$

(ii) Replace y by $\frac{c^2}{x}$

$px-\frac{c^2}{px}=c\left(p^2-\frac{1}{p^2}\right)$

$p^2x^2-cp\left(p^2-\frac{1}{p^2}\right)x-c^2=0$

Sum of roots $=\frac{cp\left(p^2-\frac{1}{p^2}\right)}{p^2}=cp-\frac{c}{p^3}$

$=x_P+x_Q.$

$\therefore x_Q=-\frac{c}{p^3}$

But $-\frac{c}{p^3}=cq,\therefore q=-\frac{1}{p^3}$

Question 13

(a) $f(x)=x^x, \therefore \ln f(x) = x\ln x.$

$\frac{f'(x)}{f(x)} = \ln x + 1$

$f'(x)=0$ when $\ln x = -1, \therefore x = \frac{1}{e}$

x	$\frac{1}{e^2}$	$\frac{1}{e}$	1
$f'(x)$	$-e^{-\frac{2}{e^2}}$	0	1
	↘	↔	↗

$\therefore$ When $x=\frac{1}{e}$, $f(x)$ is minimum.

(b) Let $\angle BPC = \angle BCP = \alpha$.

$\angle ABC = 2\alpha$ (exterior angle in a Δ = sum of 2 opposite interior angles)

$\angle BCA = 90°$ (semi-circle angle)

$\angle BAC = \alpha$ (angles in alternate segments)

$\therefore \alpha = 30°$ (angle sum in ΔABC)

$\angle ACQ = \angle CAQ = \angle ABC = 60°$ (angles in alternate segments), $\therefore \angle AQC = 60°$ (angle sum in ΔAQC)

$\angle CQO = \frac{1}{2}\angle CQA$ ($\Delta OCQ \equiv \Delta OAQ$ (SSS))

$= 30° = \angle BPC$.

$\therefore \Delta OPQ$ is isosceles, $\therefore OP = OQ$

(c) (i) $\tan\angle BAC = \frac{3}{4}, \therefore \sin\angle BAC = \frac{3}{5}, \cos\angle BAC = \frac{4}{5}$

Resolve the forces at C,

vert, $\frac{4}{5}T_1 - Mg = 0, \therefore T_1 = \frac{5Mg}{4} = \frac{50M}{4}$

hor, $\frac{3}{5}T_1 + T_2 = 0.3M\omega^2$

Sub $T_1 = \frac{50M}{4}$ gives

$T_2 = 0.3M\omega^2 - \frac{30M}{4} = 0.3M\left(\omega^2 - \frac{100}{4}\right)$

$= 0.3M\left(\omega^2 - 25\right)$

(ii) $T_2 > T_1$ gives $0.3M\left(\omega^2-25\right) > \frac{50M}{4}$

$\omega^2 - 25 > \frac{50}{1.2}$, i.e. $\omega^2 > 25 + \frac{50}{1.2} = \frac{200}{3}$

$\therefore \omega > 8.16$ rad/s

(d) (i) $p'(x) = 3ax^2 + 2bx + c.$

$\Delta = 4b^2 - 12ac = 4(b^2 - 3ac)$

$p(x)$ cuts the x-axis only once if $p'(x) > 0$

i.e. $\Delta < 0, \therefore b^2 - 3ac < 0$

(ii) $p'\left(-\frac{b}{3a}\right) = 3a\times\frac{b^2}{9a^2} + 2b\times\frac{-b}{3a} + c$

$= \frac{b^2}{3a} - \frac{2b^2}{3a} + \frac{b^2}{3a} = 0$

$p''(x) = 6ax + 2b = 6a\times\frac{-b}{3a} + 2b = 0$

$\therefore -\frac{b}{3a}$ is a root of multiplicity 3.

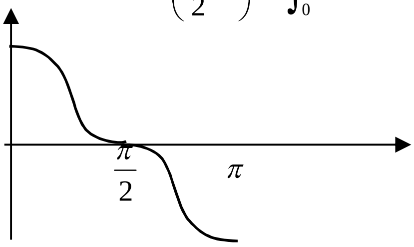
Question 14

(a) (i) $\int \sin^3 x dx = \int (1-\cos^2 x)\sin x dx$

$= \int \sin x dx - \int \cos^2 x \sin x dx$

$= -\cos x + \frac{1}{3}\cos^3 x + C$

(ii) The graph of $\cos^{2n-1} x, 0 \le x \le \pi$, is symmetrical about the point $\left(\frac{\pi}{2}, 0\right), \therefore \int_0^\pi \cos^{2n-1} x dx = 0$



(iii) $V = 2\pi \int_0^\pi xy dx = 2\pi \int_0^\pi x \sin^3 x dx$

For $I = \int_0^\pi x\sin^3 x dx$, let $u = x, dv = \sin^3 x dx$

$du = dx, v = \frac{1}{3}\cos^3 x - \cos x$

$I = \left[\frac{1}{3}x\cos^3 x - x\cos x\right]_0^\pi - \int_0^\pi \left(\frac{1}{3}\cos^3 x - \cos x\right) dx$

$= -\frac{\pi}{3} + \pi = \frac{2\pi}{3}$, since $\int_0^\pi \left(\frac{1}{3}\cos^3 x - \cos x\right) dx = 0$

$\therefore V = \frac{4\pi^2}{3}$ u^3

(b) (i) $I_0 = \int_0^1 \frac{1}{(x^2+1)^2} dx$

Let $x = \tan\theta, dx = \sec^2\theta d\theta$

When $x = 0, \theta = 0$. When $x = 1, \theta = \frac{\pi}{4}$.

$I_0 = \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{\sec^4\theta} d\theta = \int_0^{\frac{\pi}{4}} \cos^2\theta d\theta$

$= \int_0^{\frac{\pi}{4}} \frac{1+\cos 2\theta}{2} d\theta$

$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}} = \frac{\pi}{8} + \frac{1}{4}$

(ii) $I_0 + I_2 = \int_0^1 \frac{1+x^2}{(x^2+1)^2} dx = \int_0^1 \frac{1}{x^2+1} dx$

$= \left[\tan^{-1} x\right]_0^1 = \frac{\pi}{4}$

(iii) $I_4 = \int_0^1 \frac{x^4}{(x^2+1)^2} dx$

$= \int_0^1 \frac{x^4-1}{(x^2+1)^2} dx + \int_0^1 \frac{1}{(x^2+1)^2} dx$

$= \int_0^1 \frac{x^2-1}{x^2+1} dx + I_0$

$= \int_0^1 \left(1 - \frac{2}{x^2+1}\right) dx + I_0$

$= \left[x - 2\tan^{-1} x\right]_0^1 + I_0$

$= 1 - \frac{\pi}{2} + \frac{\pi}{8} + \frac{1}{4}$

$= \frac{5}{4} - \frac{3\pi}{8}$.

(c) $\text{LHS}^2 = x^3 + 2x\sqrt{x} + 1 = 2x\sqrt{x} + (x+1)(x^2 - x + 1)$

Since $(x-1)^2 \ge 0, \therefore x^2 + 1 \ge 2x$

$\therefore \text{LHS}^2 \ge 2x\sqrt{x} + (x+1)(2x - x)$

$= 2x\sqrt{x} + (x+1)x = \text{RHS}^2$.

$\therefore \text{LHS} \ge \text{RHS}$

Question 15

(a) Let $x=\alpha^2, \therefore \alpha=\sqrt{x}$

Sub $\alpha=\sqrt{x}$

$x\sqrt{x}-3\sqrt{x}+1=0$

$\sqrt{x}(x-3)=-1$

$x\left(x^2-6x+9\right)=1$

$x^3-6x^2+9x-1=0$

(b) (i) $a=\dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)=-\dfrac{\mu^2}{x^2}$

$\dfrac{1}{2}v^2=\dfrac{\mu^2}{x}+C$

When $v=0, x=b, \therefore C=-\dfrac{\mu^2}{b}$

$v^2=2\mu^2\left(\dfrac{1}{x}-\dfrac{1}{b}\right)$

$v=-\mu\sqrt{2}\sqrt{\dfrac{b-x}{bx}}$ (negative as it moves to the left)

(ii) $v=\dfrac{dx}{dt}=-\mu\sqrt{2}\sqrt{\dfrac{b-x}{bx}}$

$\int_0^t dt=-\dfrac{1}{\mu\sqrt{2}}\int_b^d\sqrt{\dfrac{bx}{b-x}}dx$

Let $x=b\cos^2\theta, dx=-2b\cos\theta\sin\theta d\theta$

When $x=b,\theta=0$. When $x=d,\theta=\cos^{-1}\sqrt{\dfrac{d}{b}}$

$t=\dfrac{2b}{\mu\sqrt{2}}\int_0^{\cos^{-1}\sqrt{\frac{d}{b}}}\dfrac{b\cos\theta}{\sqrt{b}\sin\theta}\cos\theta\sin\theta d\theta$

$=\dfrac{b\sqrt{2b}}{\mu}\int_0^{\cos^{-1}\sqrt{\frac{d}{b}}}\cos^2\theta d\theta.$

(iii) $t=\dfrac{1}{\mu}\sqrt{\dfrac{b}{2}}\left(\sqrt{bd-d^2}+b\cos^{-1}\sqrt{\dfrac{d}{b}}\right)$

When $d=0, t=\dfrac{1}{\mu}\sqrt{\dfrac{b}{2}}\times b\dfrac{\pi}{2}=\dfrac{\pi\sqrt{b^3}}{2\sqrt{2}\mu}$ s

(b) (i) $\dfrac{3!}{x(x+1)(x+2)(x+3)}=\dfrac{A}{x}+\dfrac{B}{x+1}+\dfrac{C}{x+2}+\dfrac{D}{x+3}$

where $A=\lim_{x\to 0}\dfrac{3!}{(x+1)(x+2)(x+3)}=\dfrac{3!}{6}=1$

$B=\lim_{x\to -1}\dfrac{3!}{x(x+2)(x+3)}=\dfrac{3!}{-2}=-3$

$C=\lim_{x\to -2}\dfrac{3!}{x(x+1)(x+3)}=\dfrac{3!}{2}=3$

$D=\lim_{x\to -3}\dfrac{3!}{x(x+1)(x+2)}=\dfrac{3!}{-6}=-1$

(ii) $a_k=\lim_{x\to -k}\dfrac{n!\,\cancel{(x+k)}}{x(x+1)(x+2)\ldots\cancel{(x+k)}\ldots(x+n)}$

$=\dfrac{n!}{-k(-k+1)\ldots(-k+k-1)\times(-k+k+1)\ldots(-k+n)}$

$=\dfrac{(-1)^k n!}{k(k-1)\ldots1\times1.2\ldots.(n-k)}$

$=\dfrac{(-1)^k n!}{k!(n-k)!}=(-1)^k\binom{n}{k}$

(iii) Sub $x=1$ to

$\dfrac{n!}{x(x+1)\ldots(x+n)}=\dfrac{a_0}{x}+\dfrac{a_1}{x+1}+\ldots+\dfrac{a_n}{x+n}$

$=\dfrac{1}{x}\binom{n}{0}-\dfrac{1}{x+1}\binom{n}{1}+\ldots+\dfrac{(-1)^n}{x+n}\binom{n}{n}$

$\dfrac{n!}{1.2\ldots(n+1)}=1-\dfrac{1}{2}\binom{n}{1}+\ldots+\dfrac{(-1)^n}{1+n}$

$\therefore 1-\dfrac{1}{2}\binom{n}{1}+\ldots+\dfrac{(-1)^n}{1+n}=\dfrac{n!}{(n+1)!}=\dfrac{1}{n+1}$

Question 16

(a) (i) $1=|z|=|w|, z=\cos\theta+i\sin\theta, w=\cos\alpha+i\sin\alpha$

$\text{Im}(1+z+w)=\sin\theta+\sin\alpha=0$

$\therefore \sin\theta=-\sin\alpha=\sin(-\alpha)$

$\therefore \theta=-\alpha$

$\text{Re}(1+z+w)=1+\cos\theta+\cos\alpha=0$

$\therefore 1+2\cos\theta=0$

$\cos\theta=-\dfrac{1}{2}$

$\theta=\dfrac{2\pi}{3}, \alpha=-\dfrac{2\pi}{3}$

$\therefore 1, z=\text{cis}\dfrac{2\pi}{3}$ and $w=\text{cis}\left(\dfrac{-2\pi}{3}\right)$ form an equilateral triangle.

(ii) Let $z_1=2(\cos\theta+i\sin\theta), z_2=2(\cos\alpha+i\sin\alpha)$

$\text{Re}(2i+z_1+z_2)=2(\cos\theta+\cos\alpha)=0$

$\therefore \cos\theta=-\cos\alpha$

$\therefore \theta=\pi-\alpha$

$\text{Im}(2i+z_1+z_2)=2(1+\sin\theta+\sin\alpha)=0$

$\therefore 1+2\sin\theta=0,$ since $\sin\alpha=\sin(\pi-\theta)=\sin\theta$

$\sin\theta=-\dfrac{1}{2}$

$\theta=-\dfrac{\pi}{6}, \alpha=\dfrac{7\pi}{6}$

$\therefore 2i, z_1=2\,\text{cis}\left(\dfrac{-\pi}{6}\right)$ and $z_2=2\,\text{cis}\dfrac{7\pi}{6}$ form an equilateral triangle

(b) (i) If $0, u$ and w form. the vertices of an equilateral triangle, then $u\,\text{cis}\left(\pm\dfrac{\pi}{3}\right)=w$

$u^3\,\text{cis}(\pm\pi)=w^3$

$-u^3=w^3$

$u^3+w^3=0$

$(u+w)(u^2-uw+w^2)=0$

$\therefore u^2+w^2=uw,$ since $u\neq -w$

(ii) For example, $u=\sqrt{3}+i, v=\sqrt{3}-i$

(c) (i) If Tom and another person have their own hats then there are $D(n-2)$ ways of the rest selecting the wrong hats. But instead of Tom, it can be any of other $n-1$ people, $\therefore$ the number of ways in which 2 are correct is $(n-1)D(n-2)$.

(ii) If only Tom has his own hat, then the number of derangement is $D(n-1)$, but instead of Tom, it can be any of the other $n-1$ people, $\therefore$ the number of ways in which 1 is correct is $(n-1)D(n-1)$

$\therefore D(n)=(n-1)\left[D(n-1)+D(n-2)\right]$

(iii) $D(n)-nD(n-1)=-D(n-1)+(n-1)D(n-2)$

$=-\left[D(n-1)-(n-1)D(n-2)\right]$

(iv) $D(3)-3D(2)=-\left[D(2)-2D(1)\right]=-\left[1-0\right]=-1$

$D(4)-4D(3)=-\left[D(3)-3D(2)\right]=-\left[-1\right]=1$

$D(5)-5D(4)=-\left[D(4)-4D(3)\right]=-\left[1\right]=-1$

and so on.

$\therefore D(n)-nD(n-1)=(-1)^n$.

(iv) Let $n=1, D(1)=1!\sum_{r=0}^{1}\dfrac{(-1)^r}{r!}=1-1=0, \therefore$ it's true for $n=1$.

Assume $D(n)=n!\sum_{r=0}^{n}\dfrac{(-1)^r}{r!}$ is true.

RTP $D(n+1)=(n+1)!\sum_{r=0}^{n+1}\dfrac{(-1)^r}{r!}$

$D(n+1)=(n+1)D(n)+(-1)^{n+1}$, from part (iii)

$=(n+1)n!\sum_{r=0}^{n}\dfrac{(-1)^r}{r!}+(-1)^{n+1}$

$=(n+1)!\left(1-1+\dfrac{1}{2!}-\dfrac{1}{3!}+\ldots+\dfrac{(-1)^n}{n!}\right)$

$\quad+(-1)^{n+1}\times\dfrac{(n+1)!}{(n+1)!}$

$=(n+1)!\sum_{r=0}^{n+1}\dfrac{(-1)^r}{r!}$.

By the principle of Induction it's true for all $n\geq 1$